

Lunin-Maldacena deformations with three parameters

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ABSTRACT: We examine the solution generating symmetries by which Lunin and Maldacena have generated the gravity duals of β -deformations of certain field theories. We identify the $O(2, 2, \mathbb{R})$ matrix, which acts on the background matrix $E = g + B$, where g and B are the metric and the B-field of the undeformed background, respectively. This simplifies the calculations and makes some features of the deformed backgrounds more transparent. We also find a new three-parameter deformation of the Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$. Following the recent literature on the three-parameter deformation of $AdS_5 \times S^5$, one would expect that our new solutions should correspond to non-supersymmetric marginal deformations of the relevant dual field theories.

KEYWORDS: String Duality, Gauge-gravity correspondence.

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1. Introduction

In a recent paper, Lunin and Maldacena studied a specific type of deformation of certain field theories and presented a method to find the gravity duals of the deformations [1]. The type of deformations they consider apply to both conformal and non-conformal field theories and is such that the resulting theory preserves a global $U(1) \times U(1)$ symmetry. The effect of the deformation in the Lagrangian is to introduce phases depending on the order of fields charged under this global symmetry. Such deformations are sometimes called β deformations, as they are characterized by a complex parameter, usually called β . In the particular case when β is real, they are usually called γ deformations. In this paper, we will be interested only in real deformations. The prescription Lunin and Maldacena gave works when the global $U(1) \times U(1)$ is realized geometrically on the gravity side. In this case, the geometry of the gravity background contains a 2-torus and the new gravity solutions corresponding to the deformed field theory are generated by using an $SL(2, \mathbb{R})$ symmetry associated with the 2-torus. However, this $SL(2, \mathbb{R})$ is not to be confused with the geometric $SL(2, \mathbb{R})$ acting on the complex structure modulus. It rather acts on the Kähler modulus ρ , whose real and imaginary parts are the value of the B field on the 2-torus and the volume of the 2-torus, respectively. This is the non-geometric part of the symmetry group $O(2, 2, \mathbb{R})$ associated with the two commuting isometries. We will discuss this further in section 2.

Later in [2], Frolov examined the deformed gravity solutions of [1] and showed that the classical bosonic string theory in this background exhibits a Lax pair and as such, is integrable. The Bethe ansatz for the integrable spin chain was presented in [3] and the string Bethe equations were solved explicitly. Again in [2], Frolov generalized the deformed gravity solutions of [1] by introducing two more parameters and found a 3-parameter family of deformations of $AdS_5 \times S^5$, which he proposed as the gravity dual of a non-supersymmetric marginal deformation of $\mathcal{N} = 4$ SYM, at least in the large N limit. The integrable spin chain corresponding to this background and the Bethe ansatz were constructed in [4]. Further checks of AdS/CFT correspondence for the new 3-parameter family of deformations were performed in [5]. For more on β -deformations and their gravity duals, see for example [6–16]. Also see the very recent [17] for more on Lunin-Maldacena deformations with several parameters.

One of our aims in this paper is to have a closer look at the underlying symmetries by which the new gravity solutions have been generated in [1]. In particular, we will find the $O(2, 2, \mathbb{R})$ matrix acting on the background matrix $E = g + B$, so that we obtain the solutions in an easier way. This also makes some features of the new backgrounds more transparent. For example, the transformation of the coordinates and the momenta and winding charges, which Frolov utilized in writing the Lax pair for the deformed $AdS_5 \times S^5$, follows easily from the properties of the $O(2, 2, \mathbb{R})$ matrix we find.

After we identify the $O(2, 2, \mathbb{R})$ matrix in question, we are immediately led to our main purpose, which is to find the three-parameter deformations of the Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$, whose associated AdS_5 geometries are the near horizon limit of a stack of N D-branes placed at the tip of their Calabi-Yau cones. The dual field theories were found in [18] for $T^{1,1}$ and [19, 20] for $Y^{p,q}$. One would expect, following [2, 4, 5] that the new 3-parameter deformations we find here should correspond to non-supersymmetric marginal deformations of the dual field theory.

The plan of the paper is as follows. In the next section, we will examine the solution generating symmetries used in [1] and identify the $O(2, 2, \mathbb{R})$ matrix, which acts on $E = g + B$. In section 3, we will reproduce the results of [1] by the action of the matrix we find. In section 4, we first discuss the 3-parameter generalization of [2]. Then we present our new solutions, which are the 3-parameter deformations of the Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$. All our discussions until section 5 focus on the NS sector of the solutions, whereas section 5 is devoted to the discussion of the RR sector. After discussing the regularity of the solutions in section 6, we conclude with section 7.

2. A closer look at the solution generating symmetries

Perhaps the best way to study the solution generating transformations of [1] is to look at the U-duality group of the eight dimensional TypeII string theory: $SL(2, \mathbb{R}) \times SL(3, \mathbb{R})$. Obviously, we are not being rigorous by calling this solution generating group the U-duality group, as in fact the U-duality group is the discrete $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$ [21]. However, for ease of language, we will refer to the above group as the U-duality group just as we will, throughout the text, refer to the solution generating $O(d, d, \mathbb{R})$ as the T-duality group. Let

us first examine the subgroup $SO(2, 2, \mathbb{R}) \simeq SL(2, \mathbb{R})_\tau \times SL(2, \mathbb{R})_\rho$, which is the “trivial” part of the T-duality group $O(2, 2, \mathbb{R})$ acting within IIA or IIB¹. Here the subindexes τ and ρ refers to the fact that the first $SL(2, \mathbb{R})$ acts on the complex structure modulus τ of the internal 2-torus, whereas the second one acts on its Kähler modulus ρ . $SL(2, \mathbb{R})_\tau$ is the geometric part of the T-duality associated with invariance under large diffeomorphisms of the two torus on which the ten dimensional type IIB string theory has been compactified. Its action on the complex structure modulus $\tau = \tau_1 + i\tau_2$ and the background matrix $E = g + B$ is:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad E \longrightarrow AEA^t. \tag{2.1}$$

Here

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an $SL(2, \mathbb{R})$ matrix embedded in $O(2, 2, \mathbb{R})$ as

$$\begin{pmatrix} A & 0_2 \\ 0_2 & (A^t)^{-1} \end{pmatrix}, \tag{2.2}$$

so that the background matrix E transforms as in (2.1).

On the other hand, the $SL(2, \mathbb{R})_\rho$ acts on the Kähler modulus $\rho = \rho_1 + i\rho_2 = B_{12} + i\sqrt{g}$ as

$$\rho \longrightarrow \frac{a\rho + b}{c\rho + d} \tag{2.3}$$

with $ad - bc = 1$. It is important to know how $SL(2, \mathbb{R})_\rho$ is embedded into $O(2, 2, \mathbb{R})$. One can see that the the action of $SL(2, \mathbb{R})_\rho$ on the background matrix E is through the following $O(2, 2, \mathbb{R})$ matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}. \tag{2.4}$$

It is easy to check that as the background matrix transforms as [22]

$$E \longrightarrow (AE + B)(CE + D)^{-1} \equiv \frac{AE + B}{CE + D} \tag{2.5}$$

with A, B, C, D as in (2.4), $\rho_1 = B_{12}$ and $\rho_2 = \sqrt{g}$ transform as

$$\rho_1 \longrightarrow \frac{ac(\rho_1^2 + \rho_2^2) + (ad + bc)\rho_1 + bd}{c^2(\rho_1^2 + \rho_2^2) + 2dc\rho_1 + d^2}$$

¹Obviously, $O(2, 2, Z)$ also has matrices with determinant -1, including the transformation which exchanges the Kähler and the complex structure moduli. Sometimes only this transformation is referred to as being the T-duality transformation (as in [2]), and sometimes it is referred to as the non-trivial part of the T-duality group. For an extensive review of T-duality, see [22].

$$\rho_2 \longrightarrow \frac{\rho_2}{c^2(\rho_1^2 + \rho_2^2) + 2dc\rho_1 + d^2}$$

which is equivalent to (2.3).

Yet another $SL(2, \mathbb{R})$ symmetry of the IIB string theory is the S-duality, which already exists in ten dimensions and whose Z_2 part acts on the string coupling as a strong-weak coupling duality. This group, which we will denote by $SL(2, \mathbb{R})_s$ combines with the non-geometric $SL(2, \mathbb{R})_\rho$ to form the $SL(3, \mathbb{R})$ part of the U-duality group. From the point of view of type IIA string theory, this $SL(3, \mathbb{R})$ is the geometric part of the U-duality group, associated with the large diffeomorphisms of the three torus on which the eleven dimensional M-theory has been compactified. On the other hand, the geometric $SL(2, \mathbb{R})_\tau$ of IIB is mapped under T-duality to a non-geometric $SL(2, \mathbb{R})$ which acts on the Kähler modulus of Type IIA.

The new solutions of [1] are generated through the action of $SL(3, \mathbb{R})$. The action of $SL(2, \mathbb{R})_s$ generates new solutions which correspond to a β -deformation of the field theory with a complex parameter $\beta = \sigma$. In this paper, we will only be interested in real deformations.

The supergravity solutions of Lunin and Maldacena, which correspond to β -deformations with a real parameter $\beta = \gamma$ of the dual field theory are obtained by the action of $SL(2, \mathbb{R})_\rho$ on ρ as in (2.3), with the particular choice $a = d = 1, b = 0, c = \gamma$ (so that the resulting solutions are regular). From (2.4) we see that the relevant T-duality matrix which acts on the background matrix as in (2.5) is

$$T = \begin{pmatrix} 1_2 & 0_2 \\ \Gamma & 1_2 \end{pmatrix} \tag{2.6}$$

where 1_2 and 0_2 are the 2×2 identity and zero matrices respectively, whereas

$$\Gamma = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}. \tag{2.7}$$

Perhaps it is useful at this point to recall once again that this is merely a solution generating action and not a T-duality one. In fact, as was explained in [1], on the field theory side the corresponding deformation of the field theory potential is

$$\text{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \rightarrow \text{Tr}(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2).$$

When γ is integer, (2.6) becomes an element of the T-duality group $O(2, 2, Z)$ of IIB string theory and the superpotential remains the same. There is no deformation at all. Of course, this only makes sense. From the world-sheet point of view, $O(d, d, \mathbb{R})$ acts on the moduli space of 2d CFT's² (on the world-sheet of the string theory whose target space is the background in question), taking one CFT to another connected to it by an exactly marginal deformation [22]. On the field theory side, this corresponds to starting

²When the background is flat, with the topology of T^d , the moduli space is $O(d, d, \mathbb{R})/O(d, \mathbb{R}) \times O(d, \mathbb{R})$. For curved backgrounds with d commuting isometries, it is more complicated but there is still a local $O(d, d, \mathbb{R})$ action on the moduli space.

with the four dimensional $\mathcal{N} = 4$ SCFT and deforming it to an $\mathcal{N} = 1$ SCFT with γ parameterizing the deformation (for the particular example of [6]). When γ is integer, there is no deformation and the four dimensional conformal field theory remains the same, just like the discrete $O(d, d, Z)$ maps the world-sheet CFT to a physically equivalent one.

Let us have a closer look at T in (2.6). It can be factorized in terms of the generators of $O(d, d, \mathbb{R})$ in several different ways. One way is

$$T = g_{D_1} \cdot S \cdot g_{D_1}$$

where S is as in (2.2) with

$$A = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

and g_{D_i} is the matrix of the $R \rightarrow 1/R$ transformation, called the “factorized duality” in [22], along the i th direction:

$$\begin{pmatrix} 1_2 - e_i & e_i \\ e_i & 1_2 - e_i \end{pmatrix},$$

e_i being the 2×2 matrix with all entries 0 except i th entry, which is 1. So, T corresponds to a factorized duality along one leg of the 2-torus, an $SL(2, \mathbb{R})$ transformation and another factorized duality along the same leg. This is the interpretation that was used in [2]. On the other hand, T can also be factorized as

$$T = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1_2 & \Gamma \\ 0 & 1_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}. \tag{2.8}$$

This is a volume inverting factorized duality along both legs of the torus, a geometric shift of the B-field, and then dualizing back along the two legs. This factorization will generalize to the 3-parameter deformations, as we will see in section 4.

For future use, let us note how the momenta p^i and the winding charges w^i , $i = 1, \dots, d$, $d = \dim T^d$ associated with the world-sheet currents $\partial_t x^i$ and $\partial_s x^i$ transform under the action of a T-duality matrix of the type (2.6). Let Z be the $2d$ -dimensional vector

$$Z = \begin{pmatrix} w^i \\ p^i \end{pmatrix}. \tag{2.9}$$

Under the action of a generic $T \in O(d, d, \mathbb{R})$, it transforms as [22]

$$Z \longrightarrow (T^t)^{-1} Z. \tag{2.10}$$

In our case, in which $d = 2$ and T is of the form (2.6), we have

$$\begin{pmatrix} w^1 \\ w^2 \\ p^1 \\ p^2 \end{pmatrix} \longrightarrow \begin{pmatrix} w^1 - \gamma p^2 \\ w^2 + \gamma p^1 \\ p^1 \\ p^2 \end{pmatrix}. \tag{2.11}$$

3. A shortcut to the Lunin-Maldacena backgrounds

Now that we have the T-duality matrix whose action generates the new solutions we can reproduce the Lunin-Maldacena backgrounds in an easier way. Below we will demonstrate this briefly so that our discussions for the generalized case will be more transparent. In this section, we will only be concerned with the NS sector, leaving the transformation of the RR fields to section 5. We are leaving the discussion of $AdS_5 \times S^5$ to the end, as it involves an extra feature, which will lead us to our new 3-parameter deformations.

3.1 $AdS_5 \times T^{1,1}$

Our first example is $AdS_5 \times T^{1,1}$, where $T^{1,1}$ is the coset space $(SU(2) \times SU(2))/U(1)$. This is the dual of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, which arises from a stack of N D-branes at the tip of the conifold, the conic Calabi-Yau 3-fold whose base space is $T^{1,1}$ [18]. The metric of $AdS_5 \times T^{1,1}$ is

$$\frac{ds^2}{R^2} = ds_{AdS}^2 + \frac{1}{6} \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2. \quad (3.1)$$

This is clearly an S^1 bundle over $S^2 \times S^2$ with ψ parameterizing the fiber circle which winds over the two base spheres once³. The isometry group of $T^{1,1}$ is $SU(2) \times SU(2) \times U(1)$ and in particular, it has three commuting Killing vectors: $\partial/\partial\phi_1, \partial/\partial\phi_2, \partial/\partial\psi$. [1] used the first two of them in order to generate new solutions corresponding to exactly marginal real γ -deformations of the field theory. Now we will see how the action of (2.6) does indeed reproduce their result.

First note that as S^1 is nontrivially fibered over the two isometry directions ϕ_1 and ϕ_2 , we cannot simply use the matrix in (2.6) but rather should embed it in $O(3, 3, \mathbb{R})$. The rules of this were given in [23] and the result is

$$T = \begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix} \quad (3.2)$$

with 1_3 and 0_3 being 3×3 identity and zero matrices and Γ becomes:

$$\Gamma = \begin{pmatrix} 0 & -\gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.3)$$

Applying this to the background matrix⁴

$$E = R^2 \begin{pmatrix} \frac{c_1^2}{9} + \frac{s_1^2}{6} & \frac{c_1 c_2}{9} & \frac{c_1}{9} \\ \frac{c_1 c_2}{9} & \frac{c_2^2}{9} + \frac{s_2^2}{6} & \frac{c_2}{9} \\ \frac{c_1}{9} & \frac{c_2}{9} & \frac{1}{9} \end{pmatrix} \quad (3.4)$$

³When the Hopf numbers of the S^1 bundle over the base 2-spheres are p and q the above generalizes to the manifolds $T^{p,q}$.

⁴Here we use the notation of [1]: $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$.

we obtain the new background matrix $E' = g' + B' = E \cdot (\Gamma E + 1_3)^{-1}$ from which we read

$$\frac{ds^2}{R^2} = ds_{AdS}^2 + G \left[\frac{1}{6} \sum_{i=1}^2 (G^{-1} d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + \hat{\gamma}^2 \frac{s_1^2 s_2^2}{324} d\psi^2 \right]. \quad (3.5)$$

$$\frac{B}{R^2} = \hat{\gamma} G \left[\left(\frac{s_1^2 s_2^2}{36} + \frac{c_1^2 s_2^2 + c_2^2 s_1^2}{54} \right) d\phi_1 \wedge d\phi_2 + \frac{s_1^2 c_2}{54} d\phi_1 \wedge d\psi - \frac{c_1 s_2^2}{54} d\phi_2 \wedge d\psi \right] \quad (3.6)$$

Here

$$G = \det(\Gamma E + 1_3)^{-1} = \left(1 + \hat{\gamma}^2 \left(\frac{c_1^2 s_2^2 + c_2^2 s_1^2}{54} + \frac{s_1^2 s_2^2}{36} \right) \right)^{-1}, \quad \hat{\gamma} = R^2 \gamma.$$

The dilaton always transforms as⁵ [22]

$$e^{2\Phi} \longrightarrow \frac{e^{2\Phi}}{\det(\Gamma E + 1_3)} \quad (3.7)$$

so we have $e^{2\Phi'} = G e^{2\Phi}$. This is the exact same background as in [1], although presented in a different way⁶.

3.2 $AdS_5 \times Y^{p,q}$

The next backgrounds we will consider are the recently discovered Sasaki-Einstein manifolds $Y^{p,q}$ [24, 19]. The dual field theory arises from a stack of N D-branes placed at the tip of the toric Calabi-Yau cone over $Y^{p,q}$ [19, 20]. The metric is

$$\frac{ds^2}{R^2} = ds_{AdS_5}^2 + \frac{1-y}{6} (d\theta^2 + s_\theta^2 d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - c_\theta d\phi)^2 \quad (3.8)$$

$$+ w(y) [l d\alpha + f(y) (d\psi - c_\theta d\phi)]^2 \quad (3.9)$$

Here

$$w(y) = \frac{2(a-y^2)}{1-y}, \quad q(y) = \frac{a-3y^2+2y^3}{a-y^2}, \quad f(y) = \frac{a-2y+y^2}{6(a-y^2)}$$

$$l = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}.$$

So the background matrix $E = g + b = g$ is

$$E = R^2 \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}, \quad (3.10)$$

⁵We want the asymptotic value Φ_0 of the dilaton to have the same value as in the $AdS_5 \times X_5$ case, as the exactly marginal operator by which the field theory is deformed does not couple to the dilaton. As a result, $R = (4\pi e^{\Phi_0} N)^{1/4}$ of the deformed background is the same as that of the original background.

⁶Except for an overall $G^{-1/4} = e^{-\Phi'/2} / e^{-\Phi/2}$ factor, as we are presenting our results in the string frame unlike in [1], where it is presented in the Einstein frame.

where

$$\begin{aligned}
 g_{11} &= l^2 w(y), \quad g_{22} = \frac{1-y}{6} s_\theta^2 + \frac{q(y)}{9} c_\theta^2 + w(y) f(y)^2 c_\theta^2, \quad g_{12} = -l w(y) f(y) c_\theta \\
 g_{33} &= \frac{q(y)}{9} + w(y) f(y)^2, \quad g_{13} = l w(y) f(y), \quad g_{23} = -\frac{q(y)}{9} c_\theta - w(y) f(y)^2 c_\theta.
 \end{aligned} \tag{3.11}$$

We find the new background metric and the B-field from the symmetric and the antisymmetric parts of $E' = E(\Gamma E + 1_3)^{-1}$, where Γ is as in (3.3). First let us write the determinant of $G = \det(\Gamma E + 1_3)^{-1}$.

$$G = \frac{1}{1 + \hat{\gamma}^2 \Delta},$$

where

$$\Delta = \frac{2q(y)c_\theta^2 + 3(1-y)s_\theta^2}{9(1-y)}(a - y^2)l^2, \quad \hat{\gamma} = R^2 \gamma. \tag{3.12}$$

As mentioned in the previous subsection the dilaton will simply transform as $e^{2\Phi} \rightarrow G e^{2\Phi}$. Doing the matrix multiplication we also find

$$g_{ij} \longrightarrow G g_{ij}, \quad \text{except } g_{33} \tag{3.13}$$

$$g_{33} \longrightarrow G(g_{33} + \hat{\gamma}^2 e) \tag{3.14}$$

where e is the determinant of E in (3.10)

$$\begin{aligned}
 e &= \frac{\det E}{R^6} = g_{33}(g_{11}g_{22} - g_{12}^2) + 2g_{12}g_{13}g_{23} - g_{13}^2g_{22} - g_{23}^2g_{11} \\
 &= \frac{1}{27} l^2 (2y^3 - 3y^2 + a) s_\theta^2.
 \end{aligned} \tag{3.15}$$

On the other hand we find

$$\frac{B}{R^2} = B_{12} d\alpha \wedge d\phi + B_{13} d\alpha \wedge d\psi + B_{23} d\phi \wedge d\psi$$

where

$$\begin{aligned}
 B_{12} &= \hat{\gamma}(g_{11}g_{22} - (g_{12})^2) = \hat{\gamma}\Delta, \\
 B_{13} &= \hat{\gamma}(g_{11}g_{23} - g_{12}g_{13}) = \hat{\gamma} \frac{2l^2(a + 2y^3 - 3y^2)c_\theta}{9(y-1)} \\
 B_{23} &= \hat{\gamma}(g_{12}g_{23} - g_{22}g_{13}) = -\hat{\gamma} \frac{l(a - 2y + y^2)s_\theta^2}{18}.
 \end{aligned} \tag{3.16}$$

One can check that the new background is exactly the same as that presented in [1].

3.3 $AdS_5 \times S^5$

Finally we consider the background $AdS_5 \times S^5$ whose metric is

$$\frac{ds^2}{R^2} = ds_{AdS_5}^2 + \sum_{i=1}^3 d\mu_i^2 + \mu_i^2 d\phi_i^2, \quad \text{with } \sum_i \mu_i^2 = 1. \tag{3.17}$$

When we perform the T-duality transformation with the matrix (3.2) we see that the resulting background is not the same as in [1]. After a moment's thought, one realizes that the T-duality matrix, which generates the deformed solution in [1] is in fact

$$T = \begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix} \tag{3.18}$$

where

$$\Gamma = \begin{pmatrix} 0 & -\gamma & \gamma \\ \gamma & 0 & -\gamma \\ -\gamma & \gamma & 0 \end{pmatrix} \tag{3.19}$$

Of course, first doing a coordinate transformation, then a T-duality of the form (3.2) and an inverse coordinate transformation to the original coordinates will give a T-duality transformation whose matrix is

$$\begin{pmatrix} 1_3 & 0_3 \\ A^t \Gamma A & 1_3 \end{pmatrix}. \tag{3.20}$$

So, with an appropriate choice of coordinate transformations, one can obtain a T-duality matrix of the form (3.18), starting from (3.2). This is what was done in [1], see for example [2].

Applying (3.18) one obtains the background in [1]:

$$ds^2 = R^2 \left[ds_{AdS_5}^2 + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G \mu_1^2 \mu_2^2 \mu_3^2 \left(\sum_i d\phi_i \right)^2 \right] \tag{3.21}$$

$$B = \hat{\gamma} R^2 G (\mu_1^2 \mu_2^2 d\phi_1 \wedge d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 \wedge d\phi_3 + \mu_3^2 \mu_1^2 d\phi_3 \wedge d\phi_1) \tag{3.22}$$

$$e^{2\Phi'} = G e^{2\Phi} \tag{3.23}$$

$$G = (1 + \hat{\gamma}^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2))^{-1}, \quad \hat{\gamma} = R^2 \gamma \tag{3.24}$$

At this point, one can readily guess how the generalizations with three parameters will arise.

4. Three parameter generalizations

4.1 Three-parameter deformation of $AdS_5 \times S^5$: Frolov's solution

As we discussed above, deformations of the $AdS_5 \times S^5$ background can be obtained by transforming with the T-duality matrix (3.18), which is also equivalent to performing a coordinate transformation, a factorized duality along one isometry direction, a geometric shift, then dualizing back along the same isometry direction and finally the inverse coordinate transformation. Frolov applied the latter on all the 2-tori embedded in the 3-torus and found a 3-parameter family of deformations of the $AdS_5 \times S^5$ background [2]. As one might guess from our previous discussions, it is equivalent to generating a new solution by the action of the T-duality matrix

$$\begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix}, \tag{4.1}$$

where Γ is now⁷

$$\begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{4.2}$$

Applying the T-duality matrix (4.1) to the background matrix of (3.17) we obtain exactly the solution of Frolov, whose NS sector is:

$$\begin{aligned} ds^2 &= R^2 [ds_{AdS_5}^2 + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + G\mu_1^2 \mu_2^2 \mu_3^2 (\sum_i \hat{\gamma}_i d\phi_i)^2] \\ B &= R^2 G (\hat{\gamma}_3 \mu_1^2 \mu_2^2 d\phi_1 \wedge d\phi_2 + \hat{\gamma}_1 \mu_2^2 \mu_3^2 d\phi_2 \wedge d\phi_3 + \hat{\gamma}_2 \mu_3^2 \mu_1^2 d\phi_3 \wedge d\phi_1) \\ e^{2\Phi'} &= G e^{2\Phi} \\ G &= (1 + (\hat{\gamma}_3^2 \mu_1^2 \mu_2^2 + \hat{\gamma}_1 \mu_2^2 \mu_3^3 + \hat{\gamma}_2 \mu_1^2 \mu_3^2))^{-1}, \quad \hat{\gamma}_i = R^2 \gamma_i. \end{aligned} \tag{4.3}$$

Before we move on to obtain the 3-parameter deformations of the toric backgrounds with the same method, let us pause for a moment and look at the solution generating matrix (4.1) more carefully. First of all, as the γ_i are now independent, the only possible factorization of (4.1) in terms of the generators of $O(d, d, \mathbb{R})$ is

$$\begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix} = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1_3 & \Gamma \\ 0 & 1_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}. \tag{4.4}$$

So, the action of (4.1) is equivalent to a factorized duality along all the isometry directions, then doing a Θ shift⁸ and finally an inverse factorized duality along all the isometry directions. So, perhaps it is convenient, in the spirit of [2], to call this a T Θ T transformation.

We also would like to discuss the transformation of the vector of momentum and winding charges, as this played a crucial role in [2]. Using (2.11) we see that the momenta are conserved and if w_i are the winding numbers of the undeformed background then the deformed background has

$$\phi_1(2\pi) - \phi_1(0) = 2\pi(w_1 + \gamma_2 J_3 - \gamma_3 J_2) \tag{4.5}$$

$$\phi_2(2\pi) - \phi_2(0) = 2\pi(w_2 + \gamma_3 J_1 - \gamma_1 J_3) \tag{4.6}$$

$$\phi_3(2\pi) - \phi_3(0) = 2\pi(w_3 + \gamma_1 J_2 - \gamma_2 J_1). \tag{4.7}$$

Here (J_1, J_2, J_3) are the conserved angular momenta⁹. This is the generalization of what was used in [2] in order to write down the Lax pair of the (1-parameter) deformation of the $AdS_5 \times S^5$ background. Another crucial information that [2] needed to write the Lax pair was the transformation of coordinates. Under a general T-duality transformation (2.5), the metric always transforms as

$$g \longrightarrow K^t g K \tag{4.8}$$

⁷Very recently, $d(d-1)/2$ parameter deformations of a background with d commuting $U(1)$ isometries was studied in [17]. We would expect that such deformed solutions should be generated by the action of an $O(d, d, \mathbb{R})$ matrix of the same form, where Γ is now a d -dimensional antisymmetric matrix with $d(d-1)/2$ independent parameters.

⁸It is standard to call the generators of $O(d, d, \mathbb{R})$ of this type Θ shifts [22].

⁹Here we use the letter J , not p , following [2], where p is used to name the world-sheet current.

where

$$K = (CE + D)^{-1}.$$

Instead of this “active transformation” one can also consider the “passive transformation” of the coordinates

$$d\phi_i \longrightarrow d\tilde{\phi}_i = K d\phi_i \tag{4.9}$$

so that the line element $d\phi^t g d\phi$ still transforms in the same way. (4.9) gives the following on-shell relations between the two coordinate systems

$$\begin{aligned} d\tilde{\phi}_1 &= G[(1 + \hat{\gamma}_1^2 \mu_2^2 \mu_3^2) d\phi_1 + (\hat{\gamma}_3 \mu_2^2 + \hat{\gamma}_1 \hat{\gamma}_2 \mu_2^2 \mu_3^2) d\phi_2 - (\hat{\gamma}_2 \mu_3^2 - \hat{\gamma}_1 \hat{\gamma}_3 \mu_2^2 \mu_3^2) d\phi_3] \\ d\tilde{\phi}_2 &= G[-(\hat{\gamma}_3 \mu_1^2 - \hat{\gamma}_1 \hat{\gamma}_2 \mu_1^2 \mu_3^2) d\phi_1 + (1 + \hat{\gamma}_2^2 \mu_1^2 \mu_3^2) d\phi_2 + (\hat{\gamma}_1 \mu_3^2 + \hat{\gamma}_2 \hat{\gamma}_3 \mu_1^2 \mu_3^2) d\phi_3] \\ d\tilde{\phi}_3 &= G[(\hat{\gamma}_2 \mu_1^2 + \hat{\gamma}_1 \hat{\gamma}_3 \mu_1^2 \mu_2^2) d\phi_1 - (\hat{\gamma}_1 \mu_2^2 - \hat{\gamma}_2 \hat{\gamma}_3 \mu_1^2 \mu_2^2) d\phi_2 + (1 + \hat{\gamma}_3^2 \mu_1^2 \mu_2^2) d\phi_3] \end{aligned} \tag{4.10}$$

where $\hat{\gamma}_i$ and G are as in (4.3). This played a crucial role in [2] in writing the Lax pair for the strings in the deformed background.

4.2 Three-parameter deformation of $T^{1,1}$ and $Y^{p,q}$: new solutions

We are finally in a position to write our new three-parameter deformations of the Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$. All we have to do is to generate new solutions by the action of the matrix (4.1). Following [2], one would expect our new solutions to correspond to non-supersymmetric marginal deformations of the dual field theory¹⁰. For the 3-parameter deformation of $AdS_5 \times S^5$, the corresponding integrable spin chain and the Bethe ansatz were constructed in [4]. Further checks of the AdS/CFT correspondence were performed in [5]. This motivates us to study the three-parameter deformations of the toric backgrounds, even though the resulting theory will be non-supersymmetric. Note that, as the form of the solution generating matrix is the same, (4.5) and (4.9) will also remain the same. Obviously, the matrix K depends on the background matrix E , therefore it will be more complicated in our case. Before we move on to the next point, it would be relevant to make a remark about the supersymmetry, or the lack thereof. Let’s have a look back at Frolov’s solution. We do not expect this solution to be supersymmetric because we have T-dualized along a U(1) direction with respect to which the Killing spinors have non-vanishing Lie derivative. However, a curious thing happens when all γ_i are integers. As was discussed before, in this case there is no deformation to the field theory, so the dual field theory is supersymmetric. Perhaps, this is not so curious, when one remembers the existence of supersymmetric string vacua for which the corresponding supergravity solution does not have any Killing spinors. This is the phenomena of “supersymmetry without supersymmetry”, which was discussed in [26, 27]. After this remark, we simply present the new backgrounds below.

¹⁰In [25], it was shown that the spectrum of string theory in the deformed (with three parameters) flat space contains tachyons. It was argued in [17] that this does not necessarily imply the instability of the string theory on the deformed $AdS_5 \times S^5$.

$AdS_5 \times T^{1,1}$:

$$\frac{ds^2}{R^2} = ds_{AdS_5}^2 + G \left[\frac{1}{6} \sum_{i=1}^2 (G^{-1} d\theta_i^2 + s_i^2 d\phi_i^2) + \frac{1}{9} (d\psi + c_1 d\phi_1 + c_2 d\phi_2)^2 + \frac{s_1^2 s_2^2}{324} (\hat{\gamma}_3 d\psi + \hat{\gamma}_1 d\phi_1 + \hat{\gamma}_2 d\phi_2)^2 \right] \quad (4.11)$$

$$\frac{B}{R^2} = G \left[\left(\hat{\gamma}_3 \left(\frac{c_1^2 s_2^2 + c_2^2 s_1^2}{54} + \frac{s_1^2 s_2^2}{36} \right) - \hat{\gamma}_2 \frac{c_2 s_1^2}{54} - \hat{\gamma}_1 \frac{c_1 s_2^2}{54} \right) d\phi_1 \wedge d\phi_2 + \frac{\hat{\gamma}_3 s_1^2 c_2 - \hat{\gamma}_2 s_1^2}{54} d\phi_1 \wedge d\psi - \frac{\hat{\gamma}_3 c_1 s_2^2 - \hat{\gamma}_1 s_2^2}{54} d\phi_2 \wedge d\psi \right] \quad (4.12)$$

$$e^{2\Phi'} = G e^{2\Phi}, \quad \hat{\gamma}_i = R^2 \gamma_i$$

$$G = \left(1 + \left(\hat{\gamma}_3^2 \left(\frac{c_1^2 s_2^2 + c_2^2 s_1^2}{54} + \frac{s_1^2 s_2^2}{36} \right) + \hat{\gamma}_2^2 \frac{s_1^2}{54} + \hat{\gamma}_1^2 \frac{s_2^2}{54} - \hat{\gamma}_3 \hat{\gamma}_2 \frac{s_1^2 c_2}{27} - \hat{\gamma}_1 \hat{\gamma}_3 \frac{c_1 s_2^2}{27} \right) \right)^{-1}$$

This solution reduces to that of [1] when $\gamma_3 = \gamma$ and $\gamma_1 = \gamma_2 = 0$.

$AdS_5 \times Y^{p,q}$: The background obtained by applying the solution generating matrix (4.1) has the metric components

$$g_{ij} \longrightarrow G(g_{ij} + \hat{\gamma}_i \hat{\gamma}_j e), \quad (4.13)$$

where e is as in (3.15) and

$$G^{-1} = 1 + \sum_i \hat{\gamma}_i^2 \Delta_i + 2\hat{\gamma}_1 \hat{\gamma}_2 K_3 + 2\hat{\gamma}_1 \hat{\gamma}_3 K_2 + 2\hat{\gamma}_2 \hat{\gamma}_3 K_1 \quad (4.14)$$

with

$$K_1 = g_{12}g_{13} - g_{11}g_{23}, \quad K_2 = g_{12}g_{23} - g_{22}g_{13}, \quad K_3 = g_{13}g_{23} - g_{12}g_{33}, \quad (4.15)$$

$$\Delta_1 = g_{22}g_{33} - (g_{23})^2, \quad \Delta_2 = g_{11}g_{33} - (g_{13})^2, \quad \Delta_3 = g_{11}g_{22} - (g_{12})^2. \quad (4.16)$$

$\Delta_3 = \Delta$ in (3.12) and K_1, K_2 were given in (3.16). On the other hand, $K_3 = 0$ and

$$\Delta_1 = \frac{(1-y)s_\theta^2}{6} \left(\frac{q(y)}{9} + w(y)f(y)^2 \right) = \frac{(2+a-6y+3y^2)s_\theta^2}{108},$$

$$\Delta_2 = \frac{l^2 w(y)q(y)}{9} = \frac{2l^2(a-3y^2+2y^3)}{9(1-y)}. \quad (4.17)$$

The metric, B-field and the dilaton of the new solution are

$$\frac{ds^2}{R^2} = G \left[ds_{AdS_5}^2 + \frac{1-y}{6} (d\theta^2 + s_\theta^2 d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - c_\theta d\phi)^2 + w(y) [l d\alpha + f(y)(d\psi - c_\theta d\phi)]^2 + \frac{l^2(2y^3 - 3y^2 + a)s_\theta^2}{27} (\hat{\gamma}_1 d\alpha + \hat{\gamma}_2 d\phi + \hat{\gamma}_3 d\psi)^2 \right], \quad (4.18)$$

$$\frac{B}{R^2} = G(B_{12}d\alpha \wedge d\phi + B_{13}d\alpha \wedge d\psi + B_{23}d\phi \wedge d\psi), \quad (4.19)$$

$$e^{2\Phi'} = G e^{2\Phi}, \quad \hat{\gamma}_i = R^2 \gamma_i, \quad (4.20)$$

where

$$\begin{aligned}
 B_{12} &= \hat{\gamma}_3 \Delta_3 + \hat{\gamma}_1 K_2 + \hat{\gamma}_2 K_1, \\
 B_{31} &= \hat{\gamma}_2 \Delta_2 + \hat{\gamma}_1 K_3 + \hat{\gamma}_3 K_1, \\
 B_{23} &= \hat{\gamma}_1 \Delta_1 + \hat{\gamma}_3 K_2 + \hat{\gamma}_2 K_3.
 \end{aligned}
 \tag{4.21}$$

As before, this solution reduces to that of [1] when $\gamma_3 = \gamma$ and $\gamma_1 = \gamma_2 = 0$.

5. Ramond-Ramond fields

So far, we have only considered the NS sector of the new solutions. In section 3, we showed that the action of the solution generating matrix (3.2) and (3.18) generates the NS sector of the Lunin-Maldacena backgrounds, which immediately ensures that the RR sectors also agree. Then in section 4, we obtained new 3-parameter deformations of the toric backgrounds. The aim of the present section is to study the RR sector of these new solutions.

Given a T-duality matrix, which acts on the NS sector as in (2.5) one can also work out the transformation of the RR fields under the action of this matrix. This was given in [28–31]. Here we will follow [31], which focuses on the T-duality transformations acting within Type IIA or Type IIB, namely those in $SO(d, d)$ with determinant 1. Surely, our matrix of interest, (4.1) is of this type.

As was already realized in [21, 32] RR fields combine with the NS 2-form field in an appropriate way to transform under the chiral spinor representation of $O(d, d)$. The details were worked out in [31]. Here we give a brief review of their results, without presenting the details.

The first step is to combine the RR potentials C_p of the Type II string theory (even forms in IIB and odd forms in IIA) with the NS 2-form field B_2 in the following way:

$$\begin{aligned}
 D_0 &\equiv C_0, & D_1 &\equiv C_1, \\
 D_2 &\equiv C_2 + B_2 \wedge D_0, & D_3 &\equiv C_3 + B_2 \wedge C_1, \\
 D_4 &\equiv C_4 + \frac{1}{2} B_2 \wedge C_2 + \frac{1}{2} B_2 \wedge B_2 \wedge C_0.
 \end{aligned}
 \tag{5.1}$$

Now introduce

$$D \equiv \sum_{p=0}^8 D_p, \quad F \equiv e^{-B_2} \sum_{p=0}^8 dD_p = \sum_{p=0}^8 F_{p+1}.
 \tag{5.2}$$

The indices run from 0 to 8, as we have also included the electromagnetic duals of the gauge potentials D_p ¹¹.

Next we introduce $2d$ fermionic operators ψ_i and $\psi^{i\dagger}$ satisfying

$$\{\psi_i, \psi^{j\dagger}\} = \delta_i^j \mathbf{1}, \quad \{\psi_i, \psi_j\} = \{\psi^{i\dagger}, \psi^{j\dagger}\} = 0, \quad (i, j = 1, \dots, d)
 \tag{5.3}$$

¹¹The electromagnetic duals D_{8-p} of D_p are the potential fields obtained by solving the field equations for the latter. This ensures that F defined as above satisfies $F_{10-p} = (-1)^{\lfloor \frac{p-1}{2} \rfloor} * F_p$ where $\lfloor \frac{p-1}{2} \rfloor$ is the first integer greater than or equal to $\frac{p-1}{2}$ [33].

with

$$(\psi_i)^\dagger = \psi^{i\dagger}.$$

We construct a 2^d dimensional Fock space spanned by

$$|\alpha\rangle = \psi^{i_1\dagger} \dots \psi^{i_n\dagger} |0\rangle \quad (n = 0, \dots, d) \quad (5.4)$$

where we have introduced the vacuum $|0\rangle$ such that $\psi_i |0\rangle = 0$ and $\langle 0 | 0\rangle = 1$. α in (5.4) is a multi-index $\alpha = (i_1, \dots, i_n)$ with $(i_1 < \dots < i_n)$. Now we would like to construct states in this Fock space corresponding to the form fields D and F in (5.2). To this end, [31] utilized the following one-to-one correspondence between the set of differential forms and the space of creation operators $\psi^{i\dagger}$ under which a differential form Ω

$$\Omega = \sum_n \frac{1}{n!} \Omega_{i_1 \dots i_n} dy^{i_1} \wedge \dots \wedge dy^{i_n} = \sum_q \sum_n \frac{1}{n!} \Omega_{i_1 \dots i_n}^{(q)} dy^{i_1} \wedge \dots \wedge dy^{i_n}$$

is mapped to the following operator

$$\mathbf{\Omega} \equiv \sum_n \frac{1}{n!} \Omega_{i_1 \dots i_n} \psi^{i_1\dagger} \dots \psi^{i_n\dagger} = \sum_q \sum_n \frac{1}{n!} \Omega_{i_1 \dots i_n}^{(q)} \psi^{i_1\dagger} \dots \psi^{i_n\dagger}.$$

On the right hand side, the index (q) indicates that the degree of Ω is q as a differential form on the non-compact space, whereas i_1, \dots, i_n are compact indices. This actually gives an isomorphism as an algebra. Now, to each differential form Ω one can construct the following state

$$|\Omega\rangle \equiv \mathbf{\Omega} |0\rangle. \quad (5.5)$$

The main result of [31] is that the states corresponding to D and F in (5.2) transform under $\text{SO}(d, d)$ as

$$|F\rangle \longrightarrow \mathbf{\Lambda} |F\rangle, \quad |D\rangle \longrightarrow \mathbf{\Lambda} |D\rangle \quad (5.6)$$

where

$$\mathbf{\Lambda} |\beta\rangle = \sum_\alpha |\alpha\rangle S_{\alpha\beta}(\Lambda)$$

and $S(\Lambda) = (S_{\alpha\beta}(\Lambda))$ is the spinor representation. Here Λ is a given $\text{SO}(d, d)$ matrix. [31] gave the explicit construction of the operators $\mathbf{\Lambda}$ corresponding to the generators of $\text{SO}(d, d)$.

Now, let us turn to our case of interest where $d = 3$ and the $\text{SO}(3, 3)$ matrix of concern is either of the form (3.2), (3.18) or (4.1). Also note that there is no B-field in the original background so the RR fields D_m defined in (5.1) are simply equal to C_m . The operator corresponding to our solution generating matrix, which acts on the Fock space, was constructed in [31] as

$$\mathbf{\Lambda} = \exp\left(\frac{1}{2} \Gamma_{mn} \psi_m \psi_n\right). \quad (5.7)$$

From the discussion given above one can see easily that the corresponding operator acting on the differential forms is

$$\mathbf{T} = \exp\left(\frac{1}{2} \Gamma_{mn} i_m i_n\right), \quad (5.8)$$

where i_m is contraction with the isometry direction $\partial/\partial y^m$.

In all the $AdS_5 \times X_5$ backgrounds that are of interest to us here, the only RR field is C_4 with field strength

$$F_5 = \frac{(16\pi N)\pi^3}{\text{Vol}(X_5)}(w_{AdS_5} + *w_{AdS_5}) \quad (5.9)$$

where w_{AdS_5} is the volume form on AdS_5 and $*$ is the Hodge dual, so that $*w_{AdS_5}$ is the volume form on X_5 . For example when $X_5 = Y^{p,q}$ we have

$$*w_{AdS_5} = \frac{l}{18}(1-y)s_\theta d\theta \wedge dy \wedge d\psi \wedge d\alpha \wedge d\phi.$$

So, the RR fields of the new background obtained by the solution generating matrix (2.6) is

$$\frac{16\pi N}{V}(1-\gamma i_\alpha i_\phi)(w_{AdS_5} + *w_{AdS_5}) = \frac{16\pi N}{V} \left((w_{AdS_5} + *w_{AdS_5}) + \frac{\gamma l}{18}(1-y)s_\theta d\theta \wedge dy \wedge d\psi \right). \quad (5.10)$$

Here we have used

$$V^{-1} = \frac{\text{Vol}(S^5)}{\text{Vol}(Y^{p,q})} = \frac{\pi^3}{\text{Vol}(Y^{p,q})} = \left[\frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]} \right]^{-1}.$$

In a suitable gauge we obtain

$$F_5 = \frac{16\pi N}{V}(w_{AdS_5} + *w_{AdS_5})$$

$$C_2 = -\frac{8\pi N\gamma l}{9V}(1-y)c_\theta dy \wedge d\psi.$$

This is the RR fields of the one-parameter deformed background of [1]¹². The generalization to our case is now obvious and we obtain

$$F_5 = \frac{16\pi N}{V}(w_{AdS_5} + *w_{AdS_5})$$

$$C_2 = -\frac{8\pi Nl}{9V}(1-y)c_\theta(\gamma_3 dy \wedge d\psi + \gamma_2 dy \wedge d\phi + \gamma_1 dy \wedge d\alpha). \quad (5.11)$$

On the other hand, for the $T^{1,1}$ case the volume form and the total volume are

$$*w_{AdS_5} = \frac{1}{108}s_{\theta_1}s_{\theta_2}d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\psi \quad \text{and} \quad \text{Vol}(T^{1,1}) = \frac{16\pi^3}{27}.$$

The results of [1] is obtained from

$$27\pi N(1-\gamma i_{\phi_1} i_{\phi_2})(w_{AdS_5} + *w_{AdS_5}) = 27\pi N \left((w_{AdS_5} + *w_{AdS_5}) + \gamma \frac{s_1 s_2}{108} d\theta_1 \wedge d\theta_2 \wedge d\psi \right), \quad (5.12)$$

so that in a suitable gauge one has

$$F_5 = 27\pi N(w_{AdS_5} + *w_{AdS_5}), \quad C_2 = -\frac{\gamma\pi N}{4}c_1 s_2 d\theta_2 \wedge d\psi. \quad (5.13)$$

¹²Note that there is a sign flip for C_2 . As has already been noted in [2] the sign conventions of [1] are different from the literature.

The generalization to the 3-parameter case is now obvious and here we present the result as

$$\begin{aligned}
 F_5 &= 27\pi N(w_{AdS_5} + *w_{AdS_5}) \\
 C_2 &= -\frac{\pi N}{4}c_1s_2(\gamma_3d\theta_2 \wedge d\psi + \gamma_2d\theta_2 \wedge d\phi_2 + \gamma_1d\theta_2 \wedge d\phi_1).
 \end{aligned}
 \tag{5.14}$$

6. Regularity of solutions

In this section, we analyze the regularity of the new solutions. As was already mentioned in the text, under a T-duality transformation of the form (2.5), the metric always transforms as

$$g \longrightarrow g' = \frac{1}{(CE + D)^t} g \frac{1}{(CE + D)}$$

so that the transformation of the volume form is

$$*\mathbf{1} = \sqrt{\det g} dy_1 \wedge dy_2 \wedge dy_3 \longrightarrow \sqrt{\det g'} dy_1 \wedge dy_2 \wedge dy_3 = G\sqrt{\det g} dy_1 \wedge dy_2 \wedge dy_3 = *'\mathbf{1} \tag{6.1}$$

Here y_i are the coordinates of the 3-torus and $*$ ' is the Hodge star operator of the deformed geometry. The form of $G = \det(CE + D)^{-1}$ for a general 3-parameter deformation, for which $D = I$ and $C = \Gamma$ in (4.1), was given in (4.14). First thing we observe is that G has no zeroes, so starting with a non-singular solution, we end up with a non-singular solution, unless G blows up at some points. In general, as Δ_i and K_i can become negative, G may have poles for some backgrounds, when γ_i are related. Let us make a case by case analysis for the examples considered here. In the case of $AdS_5 \times S^5$, $K_i = 0$ and $\Delta_i \geq 0$, so G does not have any poles in either of the one-parameter and three-parameter deformations. For one-parameter deformations of $T^{1,1}$ and $Y^{p,q}$, for which $\gamma_1 = \gamma_2 = 0$, there is no contribution from K_i , and $\Delta_3 \geq 0$, so again, G has no poles. Now consider the new three-parameter deformation of $T^{1,1}$, which we gave in (4.11). Here too, G has no poles, as can be seen easily when written in the following form:

$$G^{-1} = \frac{1}{108}(108 + 2s_{\theta_1}^2(c_{\theta_2}\gamma_3 - \gamma_2)^2 + 2s_{\theta_2}^2(c_{\theta_1}\gamma_3 - \gamma_1)^2 + 3\gamma_3^2s_{\theta_1}^2s_{\theta_2}^2).$$

For the analysis of $Y^{p,q}$, one should first notice that the range of y is fixed such that $1 - y > 0$, $a - y^2 > 0$, $w(y) > 0$, $q(y) \geq 0$ [24]. From this, one immediately sees that $\Delta_i \geq 0$. However, K_1 and K_2 can become negative at some points. For example, $K_1 < 0$ when $\pi/2 < \theta < \pi$. So, for some points or on some lines or planes in the parameter space, G might have poles at some values of θ and y . Note that the plane $\gamma_3 = 0$ is safe in the parameter space, as then there is no contribution from K_1 or K_2 .

The next thing to be noticed about the new three-parameter deformations found here is that it does not reduce to the undeformed solution when the volume form of the three-torus goes to zero. As can be seen from (4.13), when $e = 0$, the metric scales as $g_{ij} \rightarrow Gg_{ij}$ and the B-field does not vanish. However, when the volume forms of all the three 2-tori within the 3-torus are zero, that is, when $\Delta_1 = \Delta_2 = \Delta_3 = 0$ (which immediately implies that $K_1 = K_2 = K_3 = e = 0$), our solution reduces to the undeformed solution.

Finally, an important criteria for the regularity of the new solutions is that the RR field C_2 should vanish, as the internal torus shrinks to zero size [1]. In the previous section, we have seen that $F_3 = dC_2$ is schematically of the form

$$F_3 = \frac{(16\pi N)\pi^3}{\text{Vol}(X_5)} \sqrt{g(X_5)} dy_1 \wedge dy_2 \wedge \sum_{i=1,2,3} \gamma_i dy_i \tag{6.2}$$

where $\sqrt{g(X_5)} dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5$ is the volume form of X_5 and $dy_i, i = 1, 2, 3$ are the coordinates of the 3-torus of isometries, X_3 . So the behavior of C_2 as the 3-torus shrinks depends entirely on how the volume form of X_3 is related to that of X_5 . In the case of $T^{1,1}$, $\sqrt{g(X_5)} = \frac{1}{108} s_{\theta_1} s_{\theta_2} = \frac{1}{36} \sqrt{g(X_3)}$, so F_3 vanishes at the zeroes of the volume form of X_3 . At these points C_2 is pure gauge and as such it can be gauged away. However, in the case of $Y^{p,q}$

$$\sqrt{g(X_5)} = \frac{l(1-y)s_\theta}{18}, \quad \sqrt{g(X_3)} = \frac{ls_\theta}{3} \sqrt{\frac{2y^3 - 3y^2 + a}{3}}.$$

We see that C_2 does not necessarily vanish as the 3-torus of isometries shrinks to zero size.

7. Conclusions

In this paper, we have examined the solution generating symmetries used by [1] and identified the $O(2, 2, \mathbb{R})$ matrix that acts on $E = g + B$. The solution generating $O(2, 2, \mathbb{R})$ acts locally on the moduli space of two dimensional conformal field theories on the world-sheet, taking one CFT to another connected to it by an exactly marginal deformation. Correspondingly, the dual field theory on the boundary of AdS_5 is deformed by an exactly marginal operator, giving rise to β deformations of the original theory. On the other hand, the discrete $O(2, 2, \mathbb{Z})$ takes the world-sheet CFT to an equivalent one, just like the corresponding deformation of the field theory with integer β gives the same theory that one has started with. We have seen that some of the important features of the deformed gravity backgrounds follow from the properties of the $O(2, 2, \mathbb{R})$ matrix.

We found here a new 3-parameter family of deformations of the Sasaki-Einstein manifolds $T^{1,1}$ and $Y^{p,q}$. We have seen that the new solutions associated with $T^{1,1}$ are regular, whereas there are some problems with the regularity of the deformations of $Y^{p,q}$.

An obvious next step would be to find the Lax pair of these new backgrounds following the methods developed by [2]. It would also be interesting to examine the associated spin chain as was done in [4] for the $AdS_5 \times S^5$ case.

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